Integration of a pseudo 3D finite element ground water model with SWAT

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Outline

Ground water model in SWAT
- Assumptions and Limitations

Pseudo 3D FEM
- Assumptions and Limitations

Preliminary assessment of the model
- Upper Son Basin, Ganges, India
Groundwater Modelling in SWAT

SWAT simulates two types of aquifer for each sub-basin/HRU:

- Shallow Aquifer
- Deep Aquifer
Water balance in Shallow Aquifer

\[ aq_{sh,i} = aq_{sh,i-1} + w_{rchrg,sh} - Q_{gw} - w_{revap} - w_{pump,sh} \]

Where

- \( aq_{sh,i} \) = amount of water stored in the shallow aquifer on day \( i \) (mm of water)
- \( aq_{sh,i-1} \) = amount of water stored in the shallow aquifer on day \( i-1 \) (mm of water)
- \( w_{rchrg,sh} \) = amount of recharge entering the shallow aquifer on day \( i \) (mm of water)
- \( Q_{gw} \) = Groundwater flow, or base flow into main channel on day \( i \) (mm of water)
- \( w_{revap} \) = amount of water moving into the soil zone in response to water deficiencies in day \( i \) (mm of water)
- \( w_{pump,sh} \) = amount of water removed from shallow aquifer by pumping on a day \( i \) (mm of water)
Computation of Recharge

Recharge calculated in both aquifers are

$$w_{\text{rhr},i} = (1 - \exp[-1/\delta_{gw}]).w_{\text{seep}} + \exp[-1/\delta_{gw}].w_{\text{rchrg},i-1}$$

Exponential decay weighting function to simulate the delay from percolation to recharge

Where,

- $w_{\text{rhr},i}$ = amount of recharge entering the aquifers on day $i$ (mm of water)
- $\delta_{gw}$ = delay time or drainage time of the overlying geologic formations (days)
- $W_{\text{seep}}$ = total amount of water exiting the bottom of the soil profile on day $i$ (mm of water)
- $w_{\text{rchrg},i-1}$ = amount of recharge entering the aquifers on day $i-1$ (mm of water)
Maximum amount of water removed from the aquifer via REVAP

\[ W_{\text{revap,mx}} = \beta_{\text{rvp}} \cdot E_a \]

Capillary movement of water from shallow aquifer back to root zone

Where,

- \( W_{\text{revap,mx}} \) = maximum amount of water moving into soil zone in response to water deficiencies (mm of water)
- \( \beta_{\text{rev}} \) = revap coefficient
- \( E_a \) = actual evapotranspiration
Groundwater flow or Base flow

Base flow occur only if the amount of water stored in the shallow aquifer exceeds a threshold value specified by user, $aq_{shthr, q}$.

The baseflow rate is modelled using Hooghoudt’s equation.
Hooghoudt’s equation

- Hooghoudt’s found the equation for steady state response of groundwater flow to recharge is

\[ Q_{gw} = \frac{8000.K_{sat}^{2} h_{wtbl}}{L_{gw}} \]

Where,
- \( Q_{gw} = \) Groundwater flow or base flow, into the main channel on day \( i \) (mm of water)
- \( K_{sat} = \) Hydraulic conductivity of the aquifer (mm/day)
- \( L_{gw} = \) Distance from the ridge or sub basin divide for the groundwater system to the main channel (m)
- \( h_{wtbl} = \) Water table height (m)
Drain Spacing design

Source: http://climate.sdstate.edu/water/drainspacingcalculator/documentation.html
Assumptions by Hooghoudt’s

- Soil is homogeneous
- Darcy’s law is valid for the flow
- The hydraulic gradient at any point is equal to the slope of the water table above that point \((I = \frac{dy}{dx})\) and the water flows horizontally
Limitations

- 1D model
- Steady state assumption
- No interaction among HRU’s
Groundwater Modelling in SWAT

SWAT simulates two types of aquifer for each sub-basin/HRU:

- Shallow Aquifer
- Deep Aquifer

Model this in FEM
SWAT – FEM Coupling

SWAT model was coupled with a pseudo 3D finite element ground water model developed by Narasimhan and Sri Ranjan. (2000)

\[
\frac{\delta}{\delta x} \left( k(x,y)h \frac{\delta h}{\delta x} \right) + \frac{\delta}{\delta y} \left( k(x,y)h \frac{\delta h}{\delta y} \right) + Q(x,y,t) = S_y \frac{\delta h}{\delta t}
\]
Assumptions

- The flow is horizontal
  - Vertical hydraulic gradient is assumed to be zero – Dupuit’s assumptions
- The aquifer formation is primarily horizontal
- The aquifer can be anisotropic and heterogeneous in X and Y direction
Dupuit’s Assumption

Source: http://www.interpore.org/reference_material/mgfc-course/mgfcaqtr.html
Finite Element Solution

Galerkin’s formulation
- Selection of finite element shape
- Trial function
  - Linear Vs. Non-linear
- Formulation of element matrices
- Finite difference formulation for the time derivative
- Solving the system of equations using LU decompositions
Linear Triangular Finite Elements

\[ N_i = \frac{1}{2A^{(e)}} (a_i + b_i x + c_i y) \]
\[ N_j = \frac{1}{2A^{(e)}} (a_j + b_j x + c_j y) \]
\[ N_k = \frac{1}{2A^{(e)}} (a_k + b_k x + c_k y) \]

where:

\[ a_i = x_j^{(e)} y_k^{(e)} - x_k^{(e)} y_j^{(e)} \]
\[ a_j = x_k^{(e)} y_i^{(e)} - x_i^{(e)} y_k^{(e)} \]
\[ a_k = x_i^{(e)} y_j^{(e)} - x_j^{(e)} y_i^{(e)} \]
\[ b_i = y_j^{(e)} - y_k^{(e)} \]
\[ b_j = y_k^{(e)} - y_i^{(e)} \]
\[ b_k = y_i^{(e)} - y_j^{(e)} \]
\[ c_i = x_k^{(e)} - x_j^{(e)} \]
\[ c_j = x_i^{(e)} - x_k^{(e)} \]
\[ c_k = x_j^{(e)} - x_i^{(e)} \]

\[ A^{(e)} = \text{Area of element} \]
\[ A^{(e)} = \frac{1}{2} \begin{vmatrix} 1 & x_i^{(e)} & y_i^{(e)} \\ 1 & x_j^{(e)} & y_j^{(e)} \\ 1 & x_k^{(e)} & y_k^{(e)} \end{vmatrix} \]
Formulation of Element Matrices

\[
[K^{(e)}] = \int_{A} \left( D_{d_x} \frac{\partial [N]^T}{\partial x} \frac{\partial [N]}{\partial x} + D_{d_y} \frac{\partial [N]^T}{\partial y} \frac{\partial [N]}{\partial y} \right) dA \\
+ \int_{A} [N]^T \left( V_{tx} \frac{\partial [N]}{\partial x} + V_{ty} \frac{\partial [N]}{\partial y} \right) dA
\]  
(4.11)

Solving Eq. 4.11 using triangular elements gives:

\[
[K^{(e)}] = \frac{D_{d_x}^{*}}{4A^{(e)}} \begin{bmatrix}
 b_i^2 & b_i b_j & b_i b_k \\
 b_j b_i & b_j^2 & b_j b_k \\
 b_k b_i & b_k b_j & b_k^2
\end{bmatrix} + \frac{D_{d_y}^{*}}{4A^{(e)}} \begin{bmatrix}
 c_i^2 & c_i c_j & c_i c_k \\
 c_j c_i & c_j^2 & c_j c_k \\
 c_k c_i & c_k c_j & c_k^2
\end{bmatrix} + V_{tx} \frac{6}{6} \begin{bmatrix}
 b_i & b_j & b_k \\
 b_j & b_j & b_k \\
 b_k & b_j & b_k
\end{bmatrix} + V_{ty} \frac{6}{6} \begin{bmatrix}
 c_i & c_j & c_k \\
 c_i & c_j & c_k \\
 c_i & c_j & c_k
\end{bmatrix}
\]  
(4.12)

\[
[C^{(e)}] = \int_{A} R_d [N]^T [N] dA
\]

Solving Eq. 4.15 using lumped formulation for triangular elements gives:

\[
[C^{(e)}] = \frac{A^{(e)}}{3} R_{di} \begin{bmatrix}
 1 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 1
\end{bmatrix}
\]
Boundary conditions

- **Time varying Flux boundary**
  - Percolation from soil layer

- **Source or Sink**
  - Ground water pumping, REVAP and deep percolation

- **Time varying head boundaries**
  - Reservoirs

- **Conditional time varying head boundaries**
  - Connected/Disconnected river
Finite difference solution in time

Euler's Backward difference method $\theta = 1$

\[
\begin{align*}
([C] + \theta \Delta t[K])\{C\}^{(e)}_{t+\Delta t} &= ([C] - (1 - \theta)\Delta t[K])\{C\}^{(e)}_t \\
+ \Delta t((1 - \theta)\{F\}_t + \theta \{F\}_{t+1})
\end{align*}
\]

\[
\begin{align*}
\left([K] + \frac{[C]}{\Delta t}\right)\{C\}^{(e)}_{t+1} &= \frac{[C]}{\Delta t} \{C\}^{(e)}_t + \{F\}_{t+1}
\end{align*}
\]
Inputs For Coupling

- SWAT Model :: HRU == FEM Model :: Element
- HRU – ELE conversion (Percolation & Base flow)
  - Base flow when Avg ht. of water in element > Avg ht. of any node in the element
- Reach – Node conversion (Gaining, Loosing Stream, connected/disconnected stream)
- Reservoir – ELE conversion (Dry Elements and Wet Elements : based on the water level)
Study Area

- Son Basin in Ganga river

- Originates near Amarkantak in M.P. and flows north-north west

- Catchment area – 67,842 sq.km.

- Length – 784 km

- Tributaries – Rihand & North Koel

- Steep gradient – 0.35m to 0.55m / km
Watershed Delineation

- Flow-routing was done
- Upper Son
  Sub basins – 143
  HRUs - 599
- Lower Son
  Sub basins – 248
  HRUs - 1143
**IWFM Mesh Generator Tool**

- **Discretization of watershed**
  - Uses Triangle to subdivide the study area into triangular elements
  - Inputs – Basin and stream boundary
Discretized Upper and Lower Son Basin

Upper Son Basin: Discretization

Elements: 7404
Nodes: 3851

Nodes: UpperSon_elements

Kilometers

0 15 30 60 90 120

Lower Son Basin: Discretization

Elements: 14946
Nodes: 7807

Kilometers

0 25 50 100 150 200
Monthly Hydrograph of Lower Son

- FLOW_SWAT
- FLOW_SWATFEM
- Observed

Flow (cumec): 0, 500, 1000, 1500, 2000, 2500, 3000, 3500, 4000, 4500, 5000

Time: Jun-71, Dec-71, Jul-72, Jan-73, Aug-73, Feb-74, Sep-74, Apr-75
Assumptions

- The flow is horizontal
  - Vertical hydraulic gradient is assumed to be zero – Dupuit’s assumptions

- The aquifer formation is primarily horizontal

- The aquifer can be anisotropic and heterogeneous in X and Y direction
Summary

For simple cases, FEM formulation checked with analytical solution

In future:

- Application to a subbasin with higher base flow
  - Tifton watershed, Georgia
  - Upper North-Platte river, Nebraska